't Hooft anomaly matching:

UV: symmetry with generators T_α acting on left-handed spin $\frac{1}{2}$ fermions with anomalies $tr\left[\{T_\alpha, T_\beta\} T_\gamma\right] \neq 0$

IR: massless bound states transform under the same symmetry with generators $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, etc.

$$\longrightarrow tr\left[\{\mathcal{T}_\alpha, \mathcal{T}_\beta\} \mathcal{T}_\gamma\right] = tr\left[\{T_\alpha, T_\beta\} T_\gamma\right] \quad (1)$$

Example:

Suppose underlying theory contains n "flavors" of massless fermions in the defining rep. N of $SU(N)$ gauge group (asymptotically free) $\longrightarrow$ take N to be odd, so that there can be $SU(N)$-neutral boundstates

$\longrightarrow$ global symmetry:

$SU_L(n) \times SU_R(n) \times U_V(1)$

non-vanishing anomaly constants:

$SU(n)_L - SU(n)_L - U(1)_V$ and $SU(n)_R - SU(n)_R - U(1)_V$

with $D_{aL,bL,0} = D_{aR,bR,0} = N\delta_{ab}$

where a, b, etc. label $SU(n)$ generators λ_a, with $tr\{\lambda_a \lambda_b\} = \frac{1}{2}\delta_{ab}$.

For $n > 2$ also have:

$SU(n)_L - SU(n)_L - SU(n)_L$ and

$SU(n)_R - SU(n)_R - SU(n)_R$ anomalies

with values

$$D_{aL,bL,cL} = D_{aR,bR,cR} = N\, tr\left[\{\lambda_a, \lambda_b\}, \lambda_c\right]$$

Suppose that $SU_L(n) \times SU_R(n) \times U(1)$ symmetry is not spontaneously broken.

$\longrightarrow$ confinement leads to bound states of m_L and m_R elementary fermions of helicity $+\frac{1}{2}$ and $-\frac{1}{2}$, and $\bar{m}_L$ and $\bar{m}_R$ of their antiparticles, with

$$m_L + m_R - \bar{m}_L - \bar{m}_R = kN$$

(example: $k = 1$, $N = 3$, $m_L = 3$ and $m_R, \bar{m}_L, \bar{m}_R = 0$, Then $u_L^\alpha u_L^\beta d_L^\gamma \varepsilon_{\alpha\beta\gamma}$ is a $SU(3)$-neutral bound state)

$\longrightarrow$ encounter irreducible reps (r, s) of $SU(n)_L \times SU_R(n)$ where $\underbrace{n \times n \times \cdots \times n}_{m_L \text{ times}} \times \underbrace{\bar{n} \times \cdots \times \bar{n}}_{\bar{m}_L \text{ times}}$

$= \cdots + r + \cdots$

and $\underbrace{n \times n \times \cdots \times n}_{m_R\text{-times}} \times \underbrace{\bar{n} \times \cdots \times \bar{n}}_{\bar{m}_R\text{-times}} = \cdots + S + \cdots$

$U(1)_V$ quantum number $= kN$

Define $p(r,s,k) \equiv \#\,(r,s)$-rep of $SU(n)_L \times SU(n)_R$ with $U(1)_V$-quantum number kN appears

Then eq. (1) becomes:

$$(2)\ \sum_{r,s,k} p(r,s,k)\, d_s\, tr^{(r)}\left[\{T_a, T_b\} T_c\right] = N\, tr\left[\{\lambda_a,\lambda_b\}\lambda_c\right]$$

$$(3)\ \sum_{r,s,k} p(r,s,k)\, d_s\, k\, tr^{(r)}\left[\{T_a, T_b\}\right] = tr\left[\{\lambda_a,\lambda_b\}\right]$$

where $tr^{(r)}$ is the trace in the irreducible rep. r of $SU(n)$, and d_s is the dim. of rep. s of $SU(n)$, and $p(r,s,k)$ must be positive integers.

For $(\bar{r}, \bar{s}, -k)$ (c.c. rep.) the traces $tr^{(r)}\left[\{T_a, T_b\} T_c\right]$ and $k\, tr^{(r)}\left[\{T_a, T_b\}\right]$ have opposite values to those of (r,s,k).

$$\rightarrow \ell(r,s,k) = p(r,s,k) - p(\bar{r},\bar{s},-k)$$

and eqs. (2) and (3) become:

$$(4) \sum_{r,s,k>0} \ell(r,s,k)\, d_s\, tr^{(r)}\left[\{\tilde{T}_a, \tilde{T}_b\}\tilde{T}_c\right] = N\, tr\left[\{\lambda_a,\lambda_b\}\lambda_c\right]$$

$$(5) \sum_{r,s,k>0} \ell(r,s,k)\, d_s\, k\, tr^{(r)}\left[\{\tilde{T}_a, \tilde{T}_b\}\right] = tr\left[\{\lambda_a,\lambda_b\}\right],$$

Consider the case $n=2$:

no $SU(2)$-invariant out of 3 3-vectors

→ both sides of (4) vanish

defining 2-dim. of $SU(2)$ appears in any odd product of itself

→ get a solution of (5) by taking $\ell(r,s,k)=0$ except for $r=$ defining rep, $s=$ trivial, $k=1$

→ set $\ell(n,1,1)=1$

This solution is far from unique!

Systematic study:

Specialize to case $N=3$, $k=1$, and $\bar{m}_L=\bar{m}_R=0$

→ get following possibilities:

a) r is symmetric 3rd-rank $SU(n)$ tensor; s is trivial rep.

b) r is anti-sym 3rd-rank $SU(n)$ tensor; s is trivial

c) r is 3rd-rank of mixed symmetry; s trivial

d) r is sym 2nd-rank tensor; s is SU(n) vector

e) r is anti-sym. 2nd-rank-tensor;
s is SU(n) vector

f) r is SU(n) vector; s is sym. 2nd-rank tensor

g) r is SU(n) vector; s is anti-sym. 2nd-rank SU(n) tensor.

i r is trivial,

For $n>2$, eqs. (4) and (5) read:

$$\frac{1}{2}(n+3)(n+6)\ell_a + \frac{1}{2}(n-3)(n-6)\ell_b + (n^2-9)\ell_c$$
$$(6) \quad + n(n+4)\ell_d + n(n-4)\ell_e + \frac{1}{2}n^2(n+1)\ell_f + \frac{1}{2}n^2(n-1)\ell_g = 3$$

and

$$\frac{1}{2}(n+2)(n+3)\ell_a + \frac{1}{2}(n-2)(n-3)\ell_b + (n^2-3)\ell_c + n(n+2)\ell_d$$
$$(7) \quad + n(n-2)\ell_e + \frac{1}{2}n(n+1)\ell_f + \frac{1}{2}n(n-1)\ell_g = 1$$

$\rightarrow$ for n a multiple of 3, for all values of ℓ's, each term on the LHS of (6) is a multiple of 3

$\rightarrow$ impossible to satisfy eq. (6) for n a multiple of 3!

Take for example QCD:
has $SU(3)_L \times SU(3)_R \times U(1)_V$ global symmetry in UV (rotations of u, d, s quarks)

$\longrightarrow$ since eq. (4) cannot be satisfied, the symmetry <u>must</u> be broken in IR spontaneously!

§6.5 Consistency Conditions

Useful to assume that <u>all</u> symmetry currents (even global symmetries) are coupled to gauge fields. At the end, we can always take $g \to 0$ for these symmetries and return to a global symmetry

$\longrightarrow$ apart from anomalies, the effective action $\Gamma[A]$ is invariant under

background gauge field

$$A_{\beta\mu}(y) \longmapsto A_{\beta\mu}(y) + i \int d^4x \, \varepsilon_\alpha(x) \mathcal{T}_\alpha(x) A_{\beta\mu}(y)$$

where we must take

$$-i\,\mathcal{T}_\alpha(x) = -\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta A_{\alpha\mu}(x)} - C_{\alpha\beta\gamma} A_{\beta\mu}(x) \frac{\delta}{\delta A_{\gamma\mu}(x)}$$

in order to reproduce

$$\delta A^{\beta}{}_{\mu} = \partial_\mu \varepsilon^{\beta} + i\varepsilon^{\beta}(t^{A}{}_{\alpha})^{\beta}{}_{\gamma} A^{\gamma}{}_{\mu}$$

$$\overset{\shortparallel}{-iC^{\beta}{}_{\gamma\alpha}}$$

Taking anomalies into account, have

$$\mathcal{T}_\alpha(x)\Gamma[A] = G_\alpha[x;A]$$

where

$$D_\mu \langle J^\mu_\alpha(x)\rangle = -i\,G_\alpha[x;A]$$

and

$$\langle J^\mu_\alpha(x)\rangle = \frac{\delta}{\delta A_{\alpha\mu}(x)}\Gamma[A]$$

with $D_\mu = \partial_\mu - iA^\beta{}_\mu(x)(t_\beta)_e{}^m$ the gauge-covariant derivative

The commutation relations

$$[\mathcal{T}_\alpha(x), \mathcal{T}_\beta(y)] = iC_{\alpha\beta\gamma}\,\delta^4(x-y)\,\mathcal{T}_\gamma(x)$$

imply the "Wess-Zumino" consistency conditions:

$$\mathcal{T}_\alpha(x)\,G_\beta[y;A] - \mathcal{T}_\beta(y)\,G_\alpha[x;A]$$

$$= iC_{\alpha\beta\gamma}\,\delta^4(x-y)\,G_\gamma[y;A]$$